ON SOLUTIONS

OF DECISION PROBLEMS

WITH IMPERFECT RECALL

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Abstract

In this paper, I study decision theory in the presence of imperfect recall. I use an extension of the standard strategy concept for the analysis of extensive form games in order to examine the range of imperfect recall problems for which there exists an optimal solution. Optimality is assessed in terms of perfect recall problems associated to their corresponding imperfect recall problems.

Összefoglaló

Ebben a tanulmányban kísérletet teszek azon döntési problémák körének meghatározására, amelykben a döntéshozó memóriája nem tökéletes. Ezek egy részét a hagyományos stratégia fogalom segítségével nem lehet megoldani, csak egy új, kibõvített koncepció vezethet optimális döntésekhez ilyen helyzetekben. Ehhez a vizsgálathoz meghatározom azt is, hogy mit jelent egy döntési probléma megoldása, ha felejtés is lehetséges benne.

1. Introduction

When *Piccione* and *Rubinstein* [1994] recently took up the issue of how to analyze decision problems with imperfect recall, they almost had to start the discourse from the state it was left in the fifties. They set out to catalog the difficulties which may have prevented others to write on this topic. As their work reveals, the difficulties in the analysis of imperfect recall are not simply due to technical complexities or the vagueness surrounding the identity of players. Many concepts, techniques and approaches which serve as cornerstones for contemporary decision and game theory do not perform well in the presence of imperfect recall.

Their attention was limited to decision theory, as a natural first step. It could be asserted that they made five main observations about the interpretation of decision theory with imperfect recall. The first registers the need of employing behavioral strategies to solve some imperfect recall problems. This result has been already pointed out by *Isbell* [1957], but *Piccione* and *Rubinstein* identify additional ambiguities in interpreting behavioral strategies in imperfect recall contexts. Second, they point out that an instance of time inconsistency appears in several cases, which is not due to preference changes. Third, urged by the previous observation, they examine the possibility of interpreting imperfect recall problems as the interaction of several temporal selves. This, too, leaves substantial ambiguities in the analysis. Fourth, they discuss how to model the beliefs of the decision maker while he is in the middle of the problem. Finally, they consider the case when the decision maker may even forget his own strategy, and therewith yet an other set of interpretational dilemmas appears.

The analysis undertaken in this paper is grounded in the thesis that these five ambiguities are all tied to a further one, the ambiguity of the strategy concept in situations riddled with imperfect recall. The following simple example can give a partial illustration.

Figure 1-a exhibits a decision problem with imperfect recall. At the information set I_3, the decision maker forgot what the previous chance move was, something he could have known at either I_1 or I_2. Now suppose that the strategy he formed at the beginning prescribes to do L at I_3. Then if he would end up being at I_2, he should opt for O there. However, if at d_4 in I_3, he could indeed do R, then we may say that he should not take O at I_2. But a strategy prescribes the same action for each of the vertices in an information set, so he at I_2 cannot hope that later at d_4 in I_3 the right decision will be made. Then at I_2 there is a reason to change the strategy formulated at the beginning. Suppose that this was indeed possible. Then is it not the case that at I_3 he can deduce from the fact that the strategy has changed where he is exactly, at d_3 or at d_4? So can we allow for changing strategy in the middle of the problem? What can the decision maker know about his later ability to comply with such a change? None of these questions concerning what strategies in imperfect recall problems are would arise in a perfect recall context.

This paper presents a model of imperfect recall decision problems, and this model underwrites one sort of analysis. This model is comprehensive enough to address all the five ambiguities pointed out by *Piccione* and *Rubinstein*. I analyze decision problems with imperfect recall with the help of a strategy concept which is an extension of the standard one introduced by von *Neumann* and *Morgenstern* and later canonized by *Kuhn* [1953][1]. At the same time, the range of the beliefs of the decision maker is reduced in this model, no more is assumed about his epistemic abilities than that he has a capacity to recognize information sets and to carry out the instructions inscribed in the strategy which he constructed at the time he was confronted with the problem. This is helpful for the examination of problems featuring forgetting: assuming less rather than more about the cognitive abilities of the agent should only help the clarification of what is at the core of the difficulties in analyzing these sorts of situations. Then I set out to show what the employment of the extended strategy concept can achieve in our context by first establishing a benchmark to which solutions to imperfect recall problems should be compared. This entails the construction of associated perfect recall problems to each imperfect recall problems, and the identification of the best standard strategy which can provide the optimal solution to these derived problems. Then I show that optimal extended strategies can attain the optimum for a large set of problems, in the sense of inducing the outcome which the optimal standard strategy for the associated problem would be able to induce. I also report a full characterization of optimal extended strategies for an important class of problems. - The introduction of the concept of extended strategies may also be used to demarcate the boundary between imperfect recall problems as they relate to individuals as opposed to teams.

Of course, other approaches are also possible and promising. *Battigalli* ([1995], [1996]) studies time inconsistency in the context of imperfect recall problems, where he conceives of these problems as coordination

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1 I introduced this concept in my [1994)] "An Analysis of Decision Problems in Time". Since then, *Joseph Halpern* has independently identified a virtually identical concept,

games between temporal/modal selves. *Halpern* ([1995], [1996]) situates the current problem in a comprehensive framework of decision theory and intertemporal knowledge[2]. I give a partial assessment of these and other approaches in my (1996: §§ 23–28, 31–34), paper.

The rest of this paper is organized as follows. In the next section, I present a model of decision problems with imperfect recall and clarify the strategy concept by means of which I will analyze the best solution to these problems. *Section 3* describes the method of the analysis and reports results of a preliminary investigation. *Section 4* introduces associated perfect recall problems and optimal extended strategies. *Section 5* presents results about optimal solutions for certain classes of problems, and section 6 offers concluding remarks.

2. The Model

This section is devoted to the description of a formal model of decision problems, and therewith a formal model of decision problems with imperfect recall. Also, it specifies the strategy concept by means of which these decision problems may be solved.

2.1. Basic Assumptions

The analysis is restricted to one-shot decision problems, that is to those which are not instances of a recurrent set of identical problems.

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see *Halpern* ([1995], [1996]).

The description of the physical problem starts by positing the set of possible histories H in a decision problem (which has generic elements $h \in H$).[3] These histories are constructed as sequences of individual basic actions, themselves elements of the set A. So an individual history h is a sequence $(a^k)_{k=1}^{K}$, where the superscript k locates an individual action in the sequence. Thus, for example, a^k marks out the basic action a_i, where $a_i \in A$. Then we say that a_i is part of the history h. So we can regard the set A as a set of types of actions, and their occurrence in a sequence individuates them as an action token. The set of action tokens is denoted by $\overline{A}$, and thus we can also say that an action token a^k is part of a certain history h.

The set H is assumed to be finite here. It further has to meet the following two requirements. First, $0 \in H$, that is the empty sequence called the initial history is an element of H. Second, if $(a^k)_{k=1}^{K} \in H$ and $(a^k)_{k=1}^{K} \neq 0$, then $(a^k)_{k=1}^{K-1} \in H$.

Finally, if for H a $h = (a^k)_{k=1}^{K} \in H$ there is no a_i such that $(h, a_i) \in H$ then that history is called a terminal history. The set of terminal histories is denoted by Z, this set then represents all the courses of action available to the decision maker.

It seems to be useful to embed the formulation above into an other one which admits the mathematical object of a graph, more specifically a tree (for the current purposes a connected graph without cycles). In this second formulation, the basic primitive object is a finite tree $\Gamma = \langle \mathrm{H}, \overline{A} \rangle$. The vertices of this tree correspond to the elements of H, the edges correspond

[2] I would like to call attention to the possibility of employing the concept of 'signalling information set' as well; see *Thompson* [1953]. *Cf.* von *Neumann–Morgenstern* ([1944] [1947]: 51–54).

[3] This part of the presentation of the physical problem corresponds to the approach recommended by *Osborne* and *Rubinstein* ([1994]: 89–90, 200–202).

to the set of action tokens $\overline{A}$. The initial history $0 \in H$ will be represented by the root of the tree.

From this it follows that edges represent individuated actions, and two distinct edges may stand for the same action from the set A. We can naturally write $h' = (h, a_i)$, where a_i is the name of the action attached to the edge (a mathematical object) adjacent to both h and h'. Notice that histories became separate entities here, by being vertices, but the elements in the set H can be identified as sequences of actions as well. In this geometrical picture we can see a sequence of actions construing a history as the sequence of edges from the root of the tree to the history in question as well.

If for a $h = (a^k)_{k=1}^{K} \in H$ there is no a_i such that $(h, a_i) \in H$, then that history is called a terminal history. The set of terminal histories is denoted by Z, this set then represents all the courses of action available to the decision maker. Next, let the $A(h) = (a_i \,|\, (h, a_i) \in H)$,denote the set of feasible actions after history h. Then we can redefine terminal histories as histories for which $A(h)$ is empty. It is further required that $\forall h \in H \setminus Z$, $A(h)$ is non-singleton.

A player assignment function $R : H \setminus \mathrm{Z} \rightarrow \{chance, DM\}$, where DM denotes the decision maker, divides further the histories in $H \setminus \mathrm{Z}$. The interpretation of this function $R(\cdot)$ is immediate, it prescribes the action of either chance (Nature) or the decision maker after each non-terminal histories. $R(\cdot)$ essentially partitions the non-terminal histories: histories when the decision maker is on the move are elements of the set D (the set D could be called the set of decision histories, histories when chance is on the move are elements of the set C.

For each history in C there is an assignment of a (strictly positive) probability with which the feasible actions after that history could occur, and these probabilities are known to the decision maker and will be never forgotten. We do not need to formalize this further, and since no substantial role will be played by this probability assignment here we can denote these probabilities by f_c and leave them like that. Sometimes I will distinguish chance moves by the symbol α.

So the physical problem can be summarized now as a tuple $\langle H, R, f_c \rangle$. Note that this is only a shorthand for the full characterization by the tuple $\langle \Gamma, R, f_c \rangle$, or $\langle H, \bar{A}\ R, f_c \rangle$. Below, I will always use $\langle H, R, f_c \rangle$, for convenience.

The preferences of the decision maker are described by the function $u : Z \rightarrow \Re$ which attaches a utility index to each terminal history. Recalling that H is finite shows that the sidestepping of a more primitive construction of preferences by the direct positing of utility indices is natural. It will be further assumed that preferences do not change during the course of the problem.

Next we have to specify the beliefs of the decision maker. Beliefs about location within the problems are captured by the concept of information sets. Information sets are members of a partition **I** (with generic element I) on the set of decision vertices D. (Denote by $|I|$ the number of histories in a given information set I.) This then stipulates that if the decision maker is at a history h, he will not be able to distinguish among the histories which are contained in that element of **I** of which h is a member. Further, for the same reasons, the decision maker cannot be able to distinguish individual actions as identified by the history at which they have to be committed. If this was not so, then histories could be identified by the actions available. This requires that for all h and h' in an information set I, $A(h) = A(h')$. For the sake of consistency, one needs to stipulate also that a given type of action a_i cannot occur at more than one information set; that is there is no $h \in I$ and $h' \in I'$ $I \neq I'$, such that $a_i \in A(h)$ and $a_i \in A(h')$.

Thus, the description of the whole extensive form decision problem is now complete. This can be summarized by the tuple $\langle H, R, f_c\, \mathbf{I}, u \rangle$. Let us say that the tuple $\Delta = \langle H, R, \mathbf{I} \rangle$ stands for the extensive form. (Note that this definition is different from the standard one in that it omits f_c. This omission is justified by the fact that no substantial role is played by these probabilities in the current discussion.)

2.2. Definitions for Some Imperfect Recall Extensive Forms

Decision problems can be classified in terms of the properties of their extensive form. Moreover, we can define a decision problem with imperfect recall in terms of these properties. As a preparation for future analysis, we have to introduce first several clusters of auxiliary concepts.

In order to give proper definitions, we have to first introduce some auxiliary notions. Let us identify a set of relations on the object $\langle H, R, \mathbf{I} \rangle$. The first of these is the *initial subhistory* relation, denoted by P. It is defined on the set H as: $h'Ph$ if and only if when $h = (a^k)_{k=1}^{K}$, $h' = (a^k)_{k=1}^{L}$ for some

$L < K$. We also write $h' \in P(h)$. The inverse of this relation is denoted by S, and hSh' if and only if $h'Ph$. We write $h \in S(h')$ accordingly. In graph-theoretical terms, P is the predecessor relation, and S is the successor relation on H. Next, let us introduce an other relation on H, called *maximal* *initial subhistory*, denoted by p. This is defined as: $h'ph$ if and only if when $h = (a^k)_{k=1}^{K}$, $h' = (a^k)_{k=1}^{K-1}$. We also write $h' = p(h)$. The inverse of this relation is denoted by s, and hsh' if and only if $h'ph$, and we may write

$h \in s(h')$ accordingly. In graph-theoretical terms, p is the immediate predecessor relation, and s is the immediate successor relation. Finally, we will make use of a further relation, called the *subhistory* relation, denoted by Q. The definition of this invokes the fact that histories can be identified as sequences of actions. We say that $Q(h) = (a^k)_{k=1}^{L}$ is a subhistory of $h = (a^k)_{k=1}^{K}$, if two conditions are met. First, each $a^{k'}$ which is part of $(a^k)_{k=1}^{L}$ has to designate the same action a_i as some $a^{k''}$ which is part of h = $(a^k)_{k=1}^{K}$. Second, if two action tokens $a^{k'}$ and $a^{k'''}$ are part of $Q(h)$, and they correspond to $a^{k''}$ and $a^{k''''}$ in h, respectively: then $a^{k'}$ and $a^{k'''}$ preserve the same order in $Q(h)$ as $a^{k''}$ and $a^{k''''}$ had in the sequence h.

The various relations defined above should be extended for the sake of the coming analysis to the set of information sets. Due to the nature of the object $\langle H, R, \mathbf{I}\rangle$, there are several legitimate extensions. The following two are adopted. For two information sets I and I', I' precedes I, that is $I'PI$ if and only if $\exists h' \in I'$ and $\exists h \in I$ such that $h'Ph$. We can write $I' \in P(I)$, and the inverse relation S is naturally defined. Similarly, for two information sets I and I', I' immediately precedes I, that is $I'pI$ if and only if $\exists h' \in I'$ and $\exists h \in I$ such that $h'ph$. We can write $I' \in P(I)$, and, again, the inverse relation s is naturally defined. The employment of the same letter for denoting these relations between information sets as those between histories is justified by the fact that we recognize only one extension. For the remaining case of predecessor relations between histories and information sets, note that histories can be viewed as singleton information sets.

A second set of auxiliary concepts involves the idea of experience, introduced by *Osborne* and *Rubinstein*[4]. The *experience of actions* of the decision maker at history $h \in D$ is denoted by $V(h)$. It is defined as that sequence $(a^l)_{l=1}^{L}$ which is a subhistory of $h = (a^k)_{k=1}^{K}$, and is such that $\forall a^{l'}$ which is part of $(a^l)_{l=1}^{L}$, $\exists h' \in P(h) \cap D$ such that the action a_i corresponding to $a^{l'}$ is in $A(h')$. This amounts to saying that $V(h) = (a^l)_{l=1}^{L}$ is that subsequence of h which is constituted by actions made previously by the decision maker, as opposed to chance. Similarly, $W(h) = (\alpha^m)_{m=1}^{M}$ is the *chance experience* at $h \in D$. Here $(\alpha^m)_{m=1}^{M}$ is a subhistory of $(a^k)_{k=1}^{K}$, and for $\forall \alpha^{m'}$ part of $(\alpha^m)_{m=1}^{M}$, $\exists c \in P(h) \cap C$ such that $\alpha^{m'} \in A(c)$. Thus this is the subsequence of h made up of the chance moves in it. This latter concept will not be employed in the current subsection, but some use will be made of it in the subsequent discussion.

The most important concept in this cluster is the *experience* of the decision maker at h, denoted by $X(h)$. It is defined as the sequence $\left((I^{l-1}, a^l)_{l=1}^{L}, I^L\right)$ This sequence has the following properties. The elements a^l are just the elements of $V(h)$. And the elements I^l are the elements of $Y(h)$, the sequence making up the *experience of information sets*. This sequence is defined as follows. For $l < L$, I^l is such that if a^{l+1} is in (I^l, a^{l+1}) which is part of $X(h)$, and further if $a^{l+1} \in A(h')$ for some $h' \in P(h) \cap D$: then $h' \in I^l$. Finally, I^L is the information set which contains h.

Recall that an extensive form decision problem is a tuplet $\Delta = \langle H, R, f_c, \boldsymbol{I}, u\rangle$ and that H may stand for a finite tree or for a finite set of histories. Again, for our purposes, the extensive form $\Delta = \langle H, R, \boldsymbol{I}\rangle$ can suitably represent a given decision problem. It is useful to identify then a last group

[4] See *Osborne–Rubinstein* ([1994]: 203), and also *Piccione–Rubinstein* ([1994]: 9–10.)

of auxiliary concepts which refer to *subproblems* of an extensive form Δ. The first among these are the history-induced subproblems, denoted by Δ^h.[5] In Δ^h, the set of histories H^h consist of h and $\forall\ h' \in H$ such that $h' \in S(h)$. The player assignment function R^h is the projection of R on H^h. Similarly, the information partition $\boldsymbol{I}^h$ is the projection of $\boldsymbol{I}$ on H^h. Formally, $\boldsymbol{I}^h$ = $(I \in | \boldsymbol{I} \cap H^h \neq 0)$. There is further a partition $\mathbf{I}_s^h$ of immediate successors of h, a projection of **I** on the set H_s^h for which it is true that $\forall h' \in H_s^h$, $h' \in s(h)$. The second kind of subproblem is that of the information set induced (**I**-induced) subproblem, denoted by Δ^I, which is defined, with a slight abuse of notation, as $\cup_{h \in I} \Delta^h$. For a more precise definition one would have to first define the union operation on subproblems. Finally, we have the action induced (a_i-induced) subproblems, denoted by $\Delta^{a_{ii}}$ This consists of action tokens corresponding to a_i and $\cup_{\{h | \exists h' : a_i \in A(h'), (h', a_i = h\}} \Delta^h$. An additional, but related concept is *containment*. Here consider some Δ^h. Then if for some $I \in \mathbf{I}^h$ and for $\forall h' \in I$ we have $h' \in \Delta^h$, we say that I is con-tained in thatΔ^h. There are analogous concepts of containment for Δ^I and Δ^a.

Finally, define the length of a history h as $l(h) = |K|$ whenever $h = (a^k)_{k=1}^K$. An information set I is said to be multi-staged, if $\forall h, h'$ such that $h \in I$ and $h' \in I$, we have $l(h) = l(h')$.

Now we identify certain classes of extensive form decision problems. All these classes are related to properties of the extensive form $\Delta = \langle H, R, \boldsymbol{I} \rangle$.

[5] Note that the symbol Δ is used both for denoting extensive forms and subproblems, and thus is employed for the reference to somewhat dissimilar mathematical objects.

DEFINITION 1: *The following is a list of classes of decision problems in extensive form:*

- *An extensive form decision problem features* perfect information*, if each information set in* Δ *is singleton.*
- *An extensive form decision problem features* perfect recall *if for* $\forall h, h', I$ *such that* $h \in I$ *and* $h' \in I$*, we have* $X(h) = X(h')$*. Otherwise it features* imperfect recall.
- *An extensive form decision problem features* perfect recall of information sets*, if for* $\forall\, h, h', I$ *such that* $h \in I$ *and* $h' \in I$*, we have* $Y(h) = Y(h')$*.*
- *An extensive form decision problem is* multi-staged*, if each of its information sets are multi-staged.*
- *An extensive form decision problem features* cross-branch relevance*, if there exists* $I \in \Delta$ *such that* $\exists\, I' \in \Delta^I$ *which is not contained in* Δ^I *or if there exists* $c \in \Delta$ *such that* $\exists\, I' \in \Delta^c$ *which is not contained in* Δ^c*.*
- *An extensive form decision problem features* absent-mindedness*, if* $\exists\, I \in \mathbf{I}$ *and* $\exists h, h' \in I$*, such that* $h \in S(h')$*.*

Most of these concepts are adapted from earlier works and I retained the original name for them. The concept of perfect information decision problem is standard. The current definition of perfect recall is the same as in *Osborne–Rubinstein* ([1994]: 203). The concept of perfect recall of information sets and absent-mindedness appears in *Piccione–Rubinstein* ([1994]: 9–10), see *Figures 2-a* and *3* for examples of each of them. *Figure 4* shows a decision problem with cross-branch relevance.

2.3. Strategies, Extended Strategies, and Basic Interpretation

After having presented the basic model, I need to specify the means by which the decision maker may try to solve a particular imperfect recall decision problem. The standard tool for the implementation of the best course of action is a strategy, which is defined as a function $\sigma : \mathbf{I} \rightarrow A$, where the range is subject to familiar restrictions. I will refer to this concept as *simple strategy*. Denote by Σ the set of all such simple strategies. For the sake of simplicity, I assume that mixed strategies cannot be employed: this will not affect the forthcoming results.

This paper recognizes an extension of the concept of the simple strategy, which allows the updating of a current strategy during the problem. This, a new object, is called *extended strategy*, and it is defined by the function $\theta : \mathbf{I} \times \Sigma \rightarrow A \times \Sigma$ where Θ is the set of which θ is the generic element. Here the same restriction applies to A as above. Extended strategies are formed before the first move of the decision maker, and prescribe two operations for each information set. The first operation is the carrying out of instructions according to the strategy regarded as valid, the second operation is contingent on the information set and may call for the specification of a new valid strategy. This new strategy is then passed on to the next information set in which a decision is to be made and which is to be reached next, given the action just committed. There this new strategy will be regarded as valid. Finally, at the outset, an initial strategy is prescribed, which we may denote by σ_0.

The difference between the two strategy concepts can be brought out by a reference to a well-known explication of what strategies are in a perfect recall context, according to which simple strategies are individual pocket books (*Kuhn* [1953]). Now extended strategies can be conceived of as a collection of pocket books, and rule prescribing when they should be used. For further discussion of the concept of extended strategies, see my ([1996]: §§ 23–28, 31–34), paper.

An analysis in this framework admitting extended strategies could be conducted in two steps. First, in a reduced account of a decision problem, temporal/modal selves within the problem could be deprived of much of their epistemic resources, without abandoning the basic structure of the problem. In this reduced account, selves are only capable of recognizing the information set they are in, and carrying out instructions inscribed in the strategy. In an analysis of imperfect recall, assuming less rather than more about beliefs can only be helpful[6]. In a second step, one could relax the assumption that decision makers have no epistemic life within the problem, and seek the corresponding definition of extended strategies in this case. The analysis would then be completely analogous to the one for the first case. The present interpretation independently justifies why it was sufficient to describe the beliefs of the decision maker by the set of information sets. All what is assumed of their beliefs is that agents are capable of identifying information sets and of following the instructions inscribed in an extended strategy. This, finally, also fixes the interpretation of beliefs concerning what the strategy is, there is no problem with strategy recall here.

3. Preliminary Analysis

In this section, I start the formal analysis of imperfect recall decision problems. Let me first outline the strategy for this analysis. The aim is to

[6] Note that if the privilege of having epistemic states will be withdrawn from the decision maker while he is in the middle of the problem, the strategy employed by him can be characterized as part of a description of a finite automaton solving the problem.

demonstrate how to solve imperfect recall problems and to specify when such a solution requires the employment of extended strategies. So the analysis seeks to identify the conditions under which there is an optimal extended strategy for some class of imperfect recall problems. The precise meaning of 'optimal' will be specified formally in subsection 4.2, but we can already say that it deserves to be called optimal since it manages to implement the best course of action discernible at the beginning of the problem.

In this paper I will examine only one specific class of decision problems in extensive form. These problems, called *Bayesian* decision problems have either no chance vertex in them, or if they have one, then that is the root of the decision tree[7].

In order to reach the conclusions, much preliminary work has to be done. To set up a criterion for the optimality of the solution of these problems via extended strategies (the topic of subsection 4.2), I will show how to construct associated perfect recall problems to each such imperfect recall problem in subsection 4.1. In turn, this construction will make use of a characterization of relevant perfect recall decision problems, which will be presented in subsections 3.1 and 3.2 below. In subsection 3.4, I describe what optimal strategies for these perfect recall problems are and introduce a tool for their examination. So most of the forthcoming analysis in the next two sections is preparation for the results in *Section 5*, which, taken together, state that there is an optimal extended strategy for a series of important classes of imperfect recall problems. – In addition, I present a

[7] Note that this class includes the problems defined as "Bayesian" by *Piccione* and *Rubinstein* ([1994]: 18).

characterization of a class of imperfect recall problems in subsection 3.3, which prepares further work in *Section 5.*

3.1. A Characterization of Bayesian Perfect Recall Problems

In this subsection, I give a characterization of Bayesian perfect recall decision problems.

LEMMA 1: *A Bayesian perfect recall problem is multi-staged.*

Proof: Take any information set I and $h, h' \in I$. Then, by hypothesis, $V(h) = V(h')$ and further, $W(h) = W(h') = c$ or $W(h) = W(h') = 0$. Therefore $l(h) = l(h')$.

Now we proceed via an induction on the cardinality of the set C in δ.

(*i*) If C is empty, then δ is a perfect information problem and therefore each information set in δ is singleton, by *Definition 1*.

(*ii*)} If C is singleton, then we know that the chance vertex is at the root of the decision tree. Recall next the definition of the experience of actions $V(h)$ at a given history h. Now a new sequence, the experience of actions after the chance history c is constructed. This sequence $V^c(h)$ is defined as that sequence $(a^l)_{l=1}^{L}$ which is a subhistory of $h = (a^k)_{k=1}^{K}$, and is such that $\forall a^{l'}$ which is part of $(a^l)_{l=1}^{L}$, $\exists h' \in P(h) \cap D, h' \in \delta^c$ such that the action a_i corresponding to $a^{l'}$ is in $A(h')$.

First, we convert the definition of perfect recall into a suggestive simple lemma:

LEMMA 2: *In a Bayesian perfect recall problem with a chance vertex,* $\forall I$ *in* δ, $\forall h, h' \in I$: $V^c(h) = V^c(h')$.

Proof: By perfect recall, $\forall h, h' \in I : X(h) = X(h')$. Then $V(h) = V(h')$ and therefore $V^c(h) = V^c(h')$.

This simple observation should be supplemented by the following two results:

LEMMA 3: *In a Bayesian perfect recall problem with a chance vertex,* $\forall I$*, if* $I \in \boldsymbol{I}^{I'}$ *such that* I' *is in* δ*, then it is contained in* $\delta^{I'}$.

Proof: Suppose for a contradiction that there is $h \in I$, $I \in \boldsymbol{I}^{I}$, but h is not in δ^{I}. Then there also exists h' not in $\delta^{I'}$ and a_i part of h such that $a_i \in A(h')$ and for all $h'' \in I'$, $\iota(h') = \iota(h'')$. But therefore for a $h''' \in I$, $h''' \in \delta^{I'}$: $X(h) \neq X(h''')$, which violates perfect recall.

LEMMA 4: *In a Bayesian perfect recall problem with a chance vertex, if* $\exists h \in I$ *in* $\boldsymbol{I}$ *of* δ *and* a_i *is part of* h*, then* I *is contained in* δ^{a_i}.

Proof: Suppose there is $h \in I$ in $\boldsymbol{I}$ of δ, and a_i part of h, for which I is not contained in δ^{a_i}. Consider an other $h' \in I$. But then a_i is part of $V^c(h)$ but not of $V^c(h')$, which is impossible by *Lemma 2*.

3.2. Resolution of Uncertainty in Perfect Recall Problems

Let us here introduce an other set of auxiliary concepts which refer to *uncertainty resolution*. Uncertainty resolution is relative to the chance vertex $c \in C$ in some Bayesian perfect recall problem, if it exists. Define $A(c) = (\alpha \mid (c, \alpha) \in H)$. Then for $\forall I \in \{\boldsymbol{I}\}$ of δ, define $A_I(c) = (\alpha \mid (c, \alpha) \in H$ and $\exists h \in I$: $h \in \delta^{\alpha})$. Clearly, $A_I(c) \subseteq A(c)$. An information set I $\in \boldsymbol{I}$ of δ is fully uncertainty resolving if $A_I(c)$ is singleton and whenever $\exists I' \in \boldsymbol{I}$ of δ such that $I' \in P(I)$, then $A_I(c)$ is not singleton. A non-singleton information set $I \in \boldsymbol{I}$ of δ is partially uncertainty resolving if $|A_I(c)| < |A(c)|$, and whenever $\exists I' \in \boldsymbol{I}$ of δ such that $I' \in P(I)$, then $| A_I(c)| < | A_{I'}(c)|$. We also say that if an information set is either fully or partially uncertainty resolving, then it is uncertainty resolving.

Now some further characteristics can be read off the following corollaries, which may provide useful tests for the procedure of constructing perfect recall problems associated to Bayesian imperfect recall problems, to be described in subsection 4.1.

COROLLARY 1: *In a Bayesian perfect recall problem with one chance vertex, for* $\forall I, I' \in \boldsymbol{I}$ *of* δ*, if* $I \in \boldsymbol{I}^{I}{}'$, $|I| \leq |I'|$.

Proof: Recall that by *Lemma 2,* for all h, h' in $I \in \boldsymbol{I}$ of δ : $V^c(h) = V^c(h')$. That is they can differ only in their chance experience within δ. This, because of *Lemma 4*, establishes that for $\forall I \in \boldsymbol{I}$ *of* δ*,* $|I| = |A_I(c)|$. ($A_I(c)$ was defined above).

Next, recognize that $|A_I(c)| \leq |A_{I'}(c)|$.Suppose for a contradiction that on the contrary, $|A_I(c)| > |A_{I'}(c)|$. So there exists $\alpha \in A_I(c)$ which is not in $A_{I'}(c)$. Then while $\exists h \in I$ which is in δ^{α}, there is $h' \in I'$ which is not in δ^{α}. But by *Lemma 3*, each $h \in I$ is in $\delta^{I'}$, which leads to the contradiction.

Now since $|I| = |A_I(c)| \leq |A_{I'}(c)| .= |I'|$, the statement of the lemma is true.

This implies two immediate further corollaries:

COROLLARY 2: *In a Bayesian perfect recall problem with one chance vertex, if* $I \in \boldsymbol{I}_s^c$ *is singleton, then* $\forall I' \in \boldsymbol{I}'$ *is singleton.*

and

COROLLARY 3: *In a Bayesian perfect recall problem with one chance vertex, if for* $I, I' \in \boldsymbol{I}$ *of* δ*:* $|I| < |I'|$ *and* $I \in s(I')$*, then I is uncertainty resolving. If further I is singleton, then it is fully uncertainty resolving.*

The content of *Corollaries 1* to *3* can be summarized in the following statement: in a Bayesian decision problem with perfect recall and one chance vertex, no uncertainty resolving information set is forgotten again.

3.3. Characterization of Problems with Regular Imperfect Recall of Information Sets

As a digression to the main argument of the current section, but also as a preparation for future analysis in subsection 5.2, let us juxtapose to the just finished characterization an other one, that of a class of imperfect recall problems. This class is the closest conceivable to the class of Bayesian perfect recall problems, in the appropriate sense. So:

DEFINITION 2: *The class of Bayesian decision problems with perfect recall of information sets which are multi-staged and do not feature cross-branch relevance, is the class of problems with regular perfect recall of information sets.*

Already a first acquaintance with this definition delivers a set of immediate observations, which will be presented in the form of a lemma, for the sake of future reference:

LEMMA 5: For problems with regular perfect recall of information sets it is true that: $\forall I \in \boldsymbol{I}$ of δ, if $\mathrm{I} \in \boldsymbol{I}^{I'}$ such that $I' \in \mathbf{I}$ of δ, then it is contained in $\delta^{I'}$.

Proof: It follows immediately from Definitions 1 and 2.

It is important to recognize that this lemma is a counterpart of *Lemma 3* in the characterization of Bayesian perfect recall problems. Similarly, the next observation is the counterpart of the previous *Lemma 4*:

LEMMA 6: *There exist problems with regular perfect recall of information sets such that it is not true that if* $\exists h \in I$ *in* $\mathbf{I}$ *of* δ *and* a_i *is part of h, then I is contained in* δ^{a_i}.

Proof: Suppose for a contradiction that the statement is false. Then *Figure 2-a* provides a counterexample.

Now we are prepared for the same kind of inductive characterization of this class of problems as was employed in the case of perfect recall. The induction is on the cardinality of the set C.

(i) If C is empty, then while it could be the case that if $\exists h \in I$ in $\mathbf{I}$ of δ and a_i is part of h, then I is not contained in δ^{a_i}, it is always the case that if I $\mathbf{I}^{I'}$ such that $I' \in \mathbf{I}$ of δ, then it is contained in $\delta^{I'}$.

(ii) If C is singleton, the problem δ is characterized by *Lemmas 5* and *6*.

Clearly, this class can be regarded as the ``closest" to the class of Bayesian perfect recall problems in the sense that with the exception of the property recorded in *Lemma 4*, these problems match all the properties of the latter class.

3.4. Implementation Graphs

Let us return to the examination of Bayesian perfect recall problems. What is the optimal simple strategy for them? Take such a decision problem δ. The optimal strategy σ^* has to fulfill the requirement:

$$\sigma^* \in argmax U(\delta) = \sum_{z \in Z} p(z \mid \sigma)\, u(z)$$

Here $p(z \mid \sigma)$ stands for the probability that the terminal history z is reached if σ is adopted, and is derived appropriately from f_c. We make the further assumption that there is only one optimal strategy for a given problem. This assumption is innocuous in the present context, its violation would only complicate the analysis without giving sufficient new insights.

Let us denote by $\boldsymbol{I}^*$ the collection of information sets which could be reached during the implementation of the optimal strategy σ^*[8]. Similarly, if there is a chance vertex in the problem, record that fact. Then let us construct the *implementation graph* Γ^*, the graph of $\boldsymbol{I}^*$ and c (if it exists), in the following way. The vertices of Γ^* are the information sets in $\boldsymbol{I}^*$ and the chance vertex c (if it exists), while the edges are the actions prescribed by σ^* or the moves after c (if it exists). This construction leads then to a derived relation s^* on Γ^* which is defined formally below.

We draw an edge from some c to an I $\in$ $\boldsymbol{I}^*$, or Is^*c, if $\exists h \in I$ and $\exists \alpha \in A(c)$ such that $(c,\alpha) = h$. When should we draw an edge from an $I \in \boldsymbol{I}^*$ to an other vertex on Γ^*, $I' \in \boldsymbol{I}^*$; that is when is it the case that $I's^*I$? Whenever $\sigma^*(I) = a_i$, $h \in I$, and there is $h' \in I'$ such that $(h, a_i = h')$. The relations P^*, S^*, p^* are analogously defined.

Finally, let us make it clear that the vertices of Γ^* keep the labels they had on Γ. *Figure 5* illustrates the derivation of Γ^* on an example.

The following lemma describes this implementation graph.

LEMMA 7: Γ^* is a tree.

Proof: First, note that Γ^* is connected by the s^* relation. Second, there can be no cycles on this graph, since by *Lemma 2,* there is no I in δ for which $I \in S(I)$ and therefore $I \in S^*(I)$ is not possible.

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[8] It is a straightforward task to give a succinct description of the derivation of $\boldsymbol{I}^*$, but this is omitted here.

It is in terms of this graph Γ^* that we define our concepts of branching. An information set I $\in$ $\boldsymbol{I}^*$ is called a *branching information set* if $s^*(I)$ is non-singleton. The chance vertex c (if it exists) is called a *branching chance vertex* if $s^*(c)$ is non-singleton. The optimal strategy σ^* is branching, if it involves at least one branching information set or chance vertex.

LEMMA 8: *I is a branching information set if and only if its immediate successors on* Γ^* *are uncertainty resolving. And c is a branching chance vertex if and only if its immediate successors on* Γ^* *are uncertainty resolving.*

Proof: Suppose first that c on Γ^* is branching, but there is an I' among its immediate successors there which is not uncertainty resolving. So $I' \in \boldsymbol{I}_s^c$ on Γ and there is an other $I'' \in \boldsymbol{I}_s^c$ since c is branching. Then $A_{I'}(c) \neq A_{I''}(c)$ and none of them are empty. Therefore $|A_{I'}(c))| < |A(c)|$, thus I' is indeed uncertainty resolving by *Corollary 3*.

Suppose next that I on Γ^* is branching, but there is an I' among its immediate successors there which is not uncertainty resolving. Again, $I' \in \boldsymbol{I}_s^I$ on Γ and there is an other $I'' \in \boldsymbol{I}_s^I$ since I is branching. Also, $A_{I'}(c) \neq A_{I''}(c)$, by Lemma 4, and none of them are empty. Then there is an $\alpha \in A(c)$, such that $\alpha \in A_I(c)$, $\alpha \in A_{I''}(c)$, but it is not the case that $\alpha \in A_{I'}(c)$. But then $|A_{I'}(c)| < |A_I(c),|$ and thus I' is uncertainty resolving.

For the reverse, suppose first that $I' \in \boldsymbol{I}_s^c$ is uncertainty resolving, but c is not branching on Γ^*. Then $|A_{I'}(c)| < |A(c)|$. So there has to be an $\alpha \in A(c)$ such that α is not in $A_{I'}(c)$; and h in some $I'' \in \boldsymbol{I}_s^c$ such that $(c, \alpha) = h$. But then c is branching.

Suppose next that $I' \in \boldsymbol{I}_s^I$ is uncertainty resolving, but I is not branching on Γ^*. Now we know that $|I'| < |I|$. This means that $|A_{I'}(c)| < |A_I(c)|$ (see the proof of C*orollary 3*). Then there is $\alpha \in A_I(c)$ which is not in $A_{I'}(c)$ and there is $I'' \in \boldsymbol{I}_s^I$ such that $\alpha \in A_{I''}(c)$. But then I has to be branching.

4. Associated Perfect Recall Problems and Optimal Extended Strategies

The main question asked in this paper is whether extended strategies could provide an optimal solution to Bayesian imperfect recall problems. In order to be able to assess whether a candidate extended strategy is optimal or not, we have to first define perfect recall problems associated to these imperfect recall problems, to serve as the appropriate benchmark. Then the criterion of optimality can be expressed in terms of the optimal simple strategy for the associated problem.

4.1. Construction of Associated Perfect Recall Problems

So let us first introduce the concept of an associated perfect recall problem. Consider the extensive form δ of a Bayesian imperfect recall problem. Below, I show the construction of δ, the extensive form of its associated perfect recall problem. But again, we let extensive forms represent a whole problem, so that δ will stand for the whole associated perfect recall problem.

The first requirement this associated perfect recall problem has to meet is that its information partition $\hat{\boldsymbol{I}}$ is a coarsest refinement of **I** in δ which makes δ a Bayesian perfect recall problem. Note that there could be δ-s for which there is a coarsening of **I** which make it a perfect recall problem, a simple example of this is on *Figure 6*.

Further, an associated perfect recall problem has to be constructed according to the following procedure which imitates the steps in the characterization of Bayesian perfect recall problems in subsection 3.1. So:

if the problem has no chance vertex in it, then make each information set singleton. Next consider the set *s(c)* and the corresponding partition, $\boldsymbol{I}_s^c$. Refine this partition so that it meets the requirements for perfect recall problems inscribed in *Lemmas 1* to *4,* and call this partition $\hat{\boldsymbol{I}}_s^c$. Next consider $\cup_{I \in \hat{\boldsymbol{I}}_s^c} \boldsymbol{I}_s^I$ and refine its members so that they meet the appropriate requirements. Repeat this until each of the terminal histories are reached. This finishes the description of the procedure.

By its nature, the procedure forces a refinement of some of the members of the original partition. Since the original problem δ featured imperfect recall, it is guaranteed that at least one I ∈ **I** is refined. Then the resulting information partition $\hat{\boldsymbol{I}}$ will contain information sets which are proper subsets of the original information partition $\boldsymbol{I}$. Now we demand that the labels of information sets in $\hat{\boldsymbol{I}}$ mark both the original information set refined and the identity of the members of the new partition. So consider an $I_i \in \boldsymbol{I}$. Then find $\hat{I} \in \hat{\boldsymbol{I}}$ such that $\forall h \in \hat{\boldsymbol{I}}, h \in I_i$. Then relabel $\hat{I}$ as $I'_i, I''_i \in \hat{\boldsymbol{I}}$. If you can find more such $\hat{\boldsymbol{I}}$, then relabel them as I'''_i, I''''_i and so forth. Collect the findings in the subpartition $\boldsymbol{I}_i \subset \hat{\boldsymbol{I}}$. Now, clearly, there will be at least one $\boldsymbol{I}_i$ which is non-singleton. (See *Figure 1-b.*)

4.2. Optimal Extended Strategies

It remains to define what an optimal extended strategy is for an imperfect recall decision problem δ. First, construct the associated perfect recall problem δ to any decision problem δ, which may be unique. Then for δ find the optimal strategy σ^* for it, and then construct the appropriate $\hat{\boldsymbol{I}}^*$ and Γ^*.

Recall that an extended strategy for the problem δ is in essence a function $\theta : \boldsymbol{I} \times \Sigma \rightarrow A \times \Sigma$, where $\boldsymbol{I}$ is in δ = ⟨*H, R,* $\boldsymbol{I}$⟩ and Σ is the set of simple strategies. Denote by θ^* an extended strategy which can induce in δ the outcome which is induced by σ^* in δ. Note that it may not exist. This extended strategy θ^* is called then the *optimal extended strategy* for δ. Now the set $\Sigma^* \in \Sigma$ is the set of strategies which are actually involved in the domain of θ^*.

Consider now the constructs $\boldsymbol{I}^*_E$ and Γ^*_E, where the subscript *E* refers to 'extended strategy'. $\boldsymbol{I}^*_E$ is the same as $\hat{\boldsymbol{I}}^*$ except that each member $\hat{I}$ of a given $\boldsymbol{I}_i$ receives the label I_i. We say that $\hat{I}$ corresponds to I_i. This means that information sets on Γ^* get back the label they had in the original problem δ. This, *a fortiori*, defines the implementation graph Γ^*_E as well. (This is illustrated on *Figures 1-c*.)

It is enough to study the objects Γ^*_E and Γ^* in order to assess what the optimal extended strategy should be for a given problem. The main requirement for optimality is that whenever the optimal simple strategy σ^* for δ prescribes a certain action for an information set $\hat{I}$ in $\hat{\boldsymbol{I}}$, then in θ^*, the valid strategy at given information set I_i on Γ^*_E corresponding to that I_i should prescribe the same action for I_i. The main source of the difficulty in constructing optimal extended strategies is that at any given time in the course of the decision problem, there could be only one valid strategy prescribed by the optimal extended strategy and this valid strategy has to prescribe the same action for any information set with the same label I_i.

The following proposition will play a crucial role in the first two theorems in *Section 5*:

PROPOSTION 1: On Γ_E^*, no two immediate successors of a branching information set has labels referring to the same I_i in $\boldsymbol{I}$ of δ.

Proof: Note first that the information sets on $\hat{\boldsymbol{I}}^*$ have all different labels. Now on Γ^*, if the chance vertex is branching, then its immediate successors are uncertainty resolving by *Lemma 8*.

So consider that the chance vertex c on Γ_E^*, is branching, and assume that there are at least two information sets in $\boldsymbol{I}_s^c$ whose label refers to the same I_i in $\boldsymbol{I}$ of δ. This means that these two information sets $I_i^{'}$, $I_i^{''}$ on Γ_E^* are proper subsets of an I_i on the original Γ, as it has been clarified above. Recall the construction of associated perfect recall problems in subsection 4.1. We arrived at $\hat{\boldsymbol{I}}^*$ by refining $\boldsymbol{I}$. By *Lemma 1,* each information set is made multi-staged in δ^c. Now suppose that $I_i^{'}$ was separated from $I_i^{''}$ because it did not meet multi-stagedness, then it can be in $\boldsymbol{I}_s^c$ at all. Since there is no other reason why the procedure would have called for a separation of it from $I_i^{''}$, there can be no two distinct $I_i^{'}$, $I_i^{''}$ among $\boldsymbol{I}_s^c$.

Next take a branching information set I on Γ_E^*, and assume that there are at least two information sets in $\boldsymbol{I}_s^I$ whose label refers to the same I_i in $\boldsymbol{I}$ of δ. The procedure in 4.1 first makes each information set multi-staged by *Lemma 1*. Also, each information set in $\boldsymbol{I}_s^I$ are contained in δ^I by *Lemma 3.* Finally, if $\sigma^*(I) = a_i$ then each $I' \in \boldsymbol{I}_s^I$ is contained in δ^{a_i} by *Lemma 4*. And the procedure does not call for further refinement. By a reasoning analogous to the one in the previous paragraph, if $I_i^{'}$ was separated from $I_i^{''}$ because it did not meet any of the previous four requirements then it cannot be in $\boldsymbol{I}_s^I$ on Γ_E^*. If the separation was either because of not meeting multi-stagedness, or because of not meeting containment in δ^I, or finally because of not meeting containment in δ^{a_i} – then $I_i^{'}$ cannot be among the immediate successors of *I*. If the separation was because of not meeting containment in δ^{a_i} whenever $\sigma^*(I) = a_i$, then it cannot be the immediate successor of *I* on Γ_E^*. And again, since there is no other reason why the procedure would have called for a separation of $I_i^{'}$ from $I_i^{''}$, there can be no two distinct $I_i^{'}, I_i^{''}$ among $\boldsymbol{I}_s^I$.

5. Results

The aim of the formal analysis undertaken in this paper was to provide solutions for Bayesian imperfect recall decision problems. Solving such a decision problem means here the implementation of the best course of action discernible in the phase when the decision maker is confronted with problem, which amounts to the imposition of the formal criterion that an optimal extended strategy should be found to a given imperfect recall decision problem of the kind considered here. In a series of steps, we derived the associated perfect recall problem δ to each Bayesian imperfect recall problem δ. And it was suggested that it is enough to study the objects Γ_E^* and Γ^* in order to tell whether there is an optimal extended strategy for a given problem. The main requirement of optimality was clarified in subsection 4.2 above.

5.1. Existence Theorems

While it is not true that extended strategies can solve each and every Bayesian imperfect recall problem, we can still make the assertion that there is indeed an optimal extended strategy for a great many of them. So let us first consider:[9]

THEOREM 1: *There is an optimal extended strategy for each Bayesian imperfect recall decision problem which does not feature absent-mindedness.*

Proof: Consider, for a given Bayesian problem, a candidate optimal extended strategy θ^*. Suppose for a contradiction that there is an information set I_i in $\boldsymbol{I}_E^*$ on Γ_E^* for which the valid strategy at that I_i, $\sigma_v^* \in \Sigma^*$ say, prescribes a different action than what the optimal simple strategy prescribes for the corresponding information set $\hat{I}$ in $\hat{\boldsymbol{I}}$ on Γ^*. That is $\hat{I}$: $\sigma_v^*(I_i) \neq \sigma_v^*(\hat{I})$.

Consider first the case when the immediate predecessor set of I_i, $p(I_i)$, is empty. Then nothing could have prevented the formulation of an initial strategy for the optimal extended strategy at the outset, so that this initial strategy would coincide with what σ^* would prescribe for the corresponding $\hat{I}$.

Next, if there is indeed at least one immediate predecessor of I_i, and it is not the chance vertex, then call it I'. Suppose first that I' is not branching. By hypothesis, it cannot be the case that I_i and I' originate in the same information set of the original problem. So, again, nothing could have prevented the updating of the optimal extended strategy at I' so that it

[9] Notice that in the proof below the symbols I' and I_i designate two different ways of distinguishing an information set from I.

would coincide with what σ^* would prescribe for the corresponding $\hat{I}$. If the immediate predecessor is the chance vertex, and it is not branching, then, again, nothing could have prevented the formulation of an initial strategy for the optimal extended strategy at the outset, so that this initial strategy would coincide with what σ^* would prescribe for the corresponding $\hat{I}$.

So consider next the case when there is at least one immediate predecessor I' of I_i, and it is branching. From *Proposition 1* we know that no two immediate successors of I' can belong to the same $\boldsymbol{I}_j$. Therefore no immediate successor of I' other than I_i can belong $\boldsymbol{I}_i$. Note that it still could not be the case that I_i and I' originate in the same information set of the original problem. Then again, nothing could have prevented the updating of the optimal extended strategy at I' so that it would coincide with what σ^* would prescribe for the corresponding $\hat{I}$. Finally, consider the case when the immediate predecessor of I_i is the chance vertex, and it is branching. From *Proposition 1* we also know that no two immediate successors of c can belong to the same $\boldsymbol{I}_j$. Then again, nothing could have prevented the formulation of an initial strategy for the optimal extended strategy at the outset, so that this initial strategy would coincide with what σ^* would prescribe for the corresponding $\hat{I}$.

We conclude that at any information set I on Γ^*_E, the candidate optimal extended strategy θ^* could have induced a valid strategy for I, so that its prescription coincides with the prescription of σ^* for the corresponding information set on Γ^*. Thus there is an optimal extended strategy for each Bayesian imperfect recall decision problem.

The next statement concerns a subclass of Bayesian decision problems with absent-mindedness. Let us first define this subclass:[10]

DEFINITION 3: *Bayesian decision problems with absent-mindedness in which* $\forall I \in \boldsymbol{I}$, *for which* $\exists h, h' \in I$ *such that* $h' \in S(h)$, *it is true that* $|I| = 2$: *are called Bayesian decision problems with* binary absent-mindedness.

Then we have:

THEOREM 2: *There is an optimal extended strategy for each Bayesian imperfect recall decision problem with binary absent-mindedness.*

Proof: Consider, for a given Bayesian problem with binary absent-mindedness, a candidate optimal extended strategy θ^*. Suppose for a contradiction that there is an information set I_i in $\boldsymbol{I}_E^*$ on Γ_E^*, for which the valid strategy at that I_i, $\sigma_v^* \in \Sigma^*$ say, prescribes a different action than what the optimal simple strategy prescribes for the corresponding information set $\hat{I}$ in $\hat{\boldsymbol{I}}$ on Γ^*. That is $\hat{I}$: $\sigma_v^*(I_i) \neq \sigma^*(\hat{I})$.

Consider first the case when the immediate predecessor set of I_i, $p(I_i)$, is empty. Then nothing could have prevented the formulation of an initial strategy for the optimal extended strategy at the outset, so that this initial strategy would coincide with what σ^* would prescribe for the correspon-ding $\hat{I}$.

Next, if there is indeed at least one immediate predecessor of I_i, and it is not the chance vertex, then call it I'. Suppose first that I' is not branching. Now it still could be the case that I_i and I' originate in the same information set of the original problem. By *Definition 3*, there is no other label on the implementation graph which is a predecessor of I_i and would

[10] I would like to thank *Joe Halpern* for most important comments concerning the

originate in the same information set at the same time. So, again, nothing could have prevented the updating of the optimal extended strategy at *I′* so that it would coincide with what σ^* would prescribe for the corresponding $\hat{I}$. If the immediate predecessor is the chance vertex, and it is not branching, then, again, nothing could have prevented the formulation of an initial strategy for the optimal extended strategy at the outset, so that this initial strategy would coincide with what σ^* would prescribe for the corresponding $\hat{I}$.

So consider next the case when there is at least one immediate predecessor *I′* of I_i, and it is branching. From *Proposition 1* we know that no two immediate successors of *I′* can belong to the same $\boldsymbol{I}_j$. Therefore no immediate successor of *I′* other than I_i can belong $\boldsymbol{I}_i$. Also, repeat the reasoning about the cardinality of $\boldsymbol{I}_i$: it could still be the case that I_i and *I′* originate in the same information set of the original problem, but by *Definition 3*, there is no other label on the implementation graph which is a predecessor of I_i and would originate in the same information set at the same time. Then again, nothing could have prevented the updating of the optimal extended strategy at *I′* so that it would coincide with what σ^* would prescribe for the corresponding $\hat{I}$. Finally, consider the case when the immediate predecessor of I_i is the chance vertex, and it is branching. From *Proposition 1* we also know that no two immediate successors of *c* can belong to the same $\boldsymbol{I}_j$. Then again, nothing could have prevented the formulation of an initial strategy for the optimal extended strategy at the outset, so that this initial strategy would coincide with what σ^* would prescribe for the corresponding $\hat{I}$.

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following theorem.

We conclude that at any information set I on Γ_E^*, the candidate optimal extended strategy θ^* could have induced a valid strategy for I, so that its prescription coincides with the prescription of σ^* for the corresponding information set on Γ^*. Thus there is an optimal extended strategy for each Bayesian imperfect recall decision problem with binary absent-mindedness.

The above proofs are based on an indirect reasoning, in the sense that they do not deliver a direct characterization of the optimal extended strategy for given problems. For a specific example of how optimal extended strategies could solve an imperfect recall decision problem in our sense, consider the problem reported in *Section 1*. Again, *Figure 1-a* exhibits this multi-stage problem without absent-mindedness and precedence reversal, albeit note that it is not of perfect recall of information sets. Its associated perfect recall problem is indicated on *Figure 1-b*. If the initial chance move is $\{l\}$, the decision maker wants to reach gains 6; if it is $\{r\}$, he wants to get to where utility 4 is given to him. The implementation tree for this problem is exhibited on *Figure 1-c*. One optimal extended strategy prescribes the initial strategy σ_0 as follows: $\sigma_0(I_1) = \sigma_0(I_2) = \{D\}$, $\sigma_0(I_3) = \{L\}$. If the problem reaches I_1, no updating is necessary. But if it reaches I_2, then there a new strategy σ_1 should be made valid, for which: $\sigma_1(I_1) = \sigma_1(I_2) = \{D\}$, $\sigma_1(I_3) = \{R\}$. And there is an other optimal extended strategy symmetrical to the previous one.

It remains to be shown what is the range of problems which can be solved by extended strategies[11]. But note that the problems covered by the above theorems include each case mentioned in the *Piccione–Rubinstein*

[1994]. – For a Bayesian problem which cannot be solved by extended strategies, examine the example on *Figure 7*. Here no optimal extended strategy can achieve an "exit" at the right time, at h_3. This example provides a partial explanation of why we have considered only binary absent-minded problems in this subsection. But it is clear that there may be optimal extended strategies for non-binary problems as well. What is at stake is that no three labels for a given information set would appear on a branch of the implementation graph which are immediate successors of each other.

5.2. Optimal Extended Strategies for Problems with Regular Perfect Recall of Information Sets

In this subsection, we will examine the properties of the optimal extended strategies for one specific class of Bayesian problems, those with regular perfect recall of information sets. This is an important case, since it is the only identifiable class of problems which features neither cross-branch relevance, nor absent-mindedness. We have made preparations for this examination in subsection 3.3, where the extensive form of these problems were characterized. That work will now support the proof of the following statement:

LEMMA 9: No label on the implementation graph of a Bayesian problem with regular perfect recall of information sets occurs twice.

Proof: Suppose for a contradiction that there is indeed a label which appears twice on the implementation graph. Denote these by $I_i^{'}$ and $I_i^{''}$,

[11] See *Halper*n (1996) for a similar theorem which addresses the class of problems in which we do not find chance histories.

respectively. This means that the construction of the associated perfect recall problem led to the refinement of the information set I_i.

Recall first that in these problems each information set has to be multi-staged, and recognize that, trivially, I_i cannot be the root of the decision tree. Also, neither $I_i^{'}$ nor $I_i^{''}$ can be immediate successors of the chance vertex on the implementation graph, since by the characterization of these problems, I_i would not then be refined by the procedure of constructing the associated perfect recall problem (see subsection 3.3).

Note next that $I_i^{'}$ and $I_i^{''}$ cannot have the same immediate predecessor on the implementation graph by *Proposition 1*. But we can affirm then that the problem cannot be that of regular perfect recall of information sets. To see this, note that on the basis of the characterization of these problems, we can state that the only reason why I_i could have been refined in the first place by the procedure is that the condition of *Lemma 4* did not hold, that is (without loss of generality) $\exists h \in I_i^{'}$ and a_i part of h, such that $I_i^{'}$ is not contained in δ^{a_i}. But then recall that the implementation graph records the actions which are demanded by the optimal simple strategy for the associated problem. So if it was the case that $\sigma^*(I_j) = a_i'$, then only one of $I_i^{'}$ or $I_i^{''}$ could have been part of the implementation graph. So the supposition led to a contradiction.

Now we can state the existence of optimal extended strategies for this class, and their main property:

THEOREM 3: *There is an optimal extended strategy for each Bayesian problems with regular perfect recall of information sets. Further, there is no need for updating the initial strategy in that optimal extended strategy.*

This can be reformulated as the proposition that for this class, simple strategies can attain the first best.

Proof: The first part follows from the conjunction of *Theorem 1* and the fact that problems with regular perfect recall of information sets do not feature absent-mindedness by *Definition 2*.

For the second part, note that we can construct optimal extended strategies here as follows. Take the optimal simple strategy σ^* for the associated problem. Find the correspondence between the labels appearing on the implementation graph and the original information sets. This correspondence is bijective by the above *Lemma 9*. Then render the same actions for the initial strategy in the optimal extended strategy to the information sets which have counterparts on the implementation graph, as the optimal simple strategy rendered to the labels appearing on that graph. Then do not prescribe any further updating in the optimal extended strategy.

This result is useful at least for the reason that it marks out the fact that the solution of imperfect recall problems without cross-branch relevance or absent-mindedness do not require the use of generic extended strategies. Time inconsistency can be overcome by simply following the strategy constructed at the phase when the problem was originally introduced.

Finally, let us illustrate the above result by means of a simple example, shown on (see *Figures 2-a, b, c*). This problem is of perfect recall of information sets. Here the initial strategy should prescribe $\{l\}$ for the first information set and $\{L\}$ for the second; and that is it. There is no need for updating.

6. Concluding Remarks

A lot of work remains to be done before a more complete assessment of what extended strategies are able to achieve can be made. More specifically, one wishes to identify the precise range of problems for which the optimal extended strategies can achieve the first best, in the sense clarified above. This should be accompanied by an analysis of the properties of optimal extended strategies for individual classes of imperfect recall problems. Also, there is a legitimate interest in higher order extensions of the standard strategy concept, in which what have been called extended strategies could be updated as well (see *Halpern* [1995]). These could solve problems like that on *Figure 7*. These could be regarded just as much a straightforward and legitimate extension of the original concept as the one reported here. However, given their increasing complexity, their employment may overstrain the resources of a decision maker.

Finally, let me mention certain other issues which should be taken up by future studies of imperfect recall problems. First, it seems that the precise relationship between the representation of decision problems (and games) by means of histories, as in *Osborne–Rubinstein* [1994] and *Piccione–Rubinstein* [1994], and by graph-theoretical concepts should be further clarified. Second, one should devote more attention to the ideas developed in the early stages of game theory, as they relate to the problem of imperfect recall. It seems to be especially worthwhile to address the original formulation of games in von *Neumann* and *Morgenstern* ([1944], [1947]), and that of *Thompson* [1953] and *Isbell* [1957], in this regard.

Third, the connections of the current account of imperfect recall decision problems to the standard model of game theory, on the one hand, and the framework proposed by *Maskin* and *Tirole* [1994] on the other, should be also explored. Fourth, and this is most important, a full engagement with the study of game theory with imperfect recall seems to be most promising; especially since this would enable us to construct models which can capture the phenomenon of forgetting in economic situations. Finally, situations when preferences changes may interact with forgetting should be also addressed. See the preliminary remarks on this issue in my [1996: §§ 35–37], paper.

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